in the *Penny Cyclopædia* and the *Supplement* to the same work. These two have since been combined in the article under the same title in the *English Cyclopædia* (" Arts and Sciences").

(49). Till very recently, when, as I am informed, it has been introduced by Mr. Todhunter in his *Theory of Equations,* Mr. Horner's method has been entirely ignored by all the University writers. It is not for me to suggest a reason for this; but it is certainly, in the circumstances, not a little singular to find retained in the text books the antiquated and laborious methods of transformation and solution, instead of Horner's elegant and efficient processes.

On the Construction of Tables of Mortality. *By* W. S. B. Woolhouse, F.R.A.S., F.S.S., *Vice-President of the Institute of Actuaries, &c.*

[Read before the Institute, 30th April, 1866.]

A TABLE of mortality is designed to represent the number of lives which, according to the best deductions from past experience, may be expected to survive at the termination of each successive year of age, supposing these survivors to be derived exclusively from a certain number of persons originally taken either at birth or at a given age. As a practical index of the mathematical law of average mortality such a table may be taken as a trustworthy guide for the future, provided that the number of lives which enter into its formation be sufficiently large and the particulars respecting them be correctly registered; and, what is equally important, that the observations shall extend over a considerable number of years.

Whether theoretically or practically considered, a table of mortality presents, in the most simple, complete and convenient form, the elementary data requisite for all computations involving the contingencies of life, and it is, therefore, universally adopted as the primitive basis upon which the whole superstructure of life assurance is built. In fact, it is the general permanence in this progressive distribution of mortality, between narrow probable limits of divergence, that constitutes the main stability and efficiency of Life Assurance Institutions. It is consequently needless to insist on the importance of accumulating reliable facts as to the statistics of human life, whenever that can be done on a large and efficient, scale, with the view of obtaining more perfect tables, or of more strictly testing the general accuracy of those in ordinary use, upon which an enormous amount of monied transactions is necessarily made to depend.

In the year 1839 I had occasion, officially, to go into an investigation of the mortality in the Indian army. An abstract of this investigation was published at the time in the form of a pamphlet. As discussions of this kind are, as a matter of course, attended by a somewhat tedious amount of numerical operations, I was induced, in the first instance, to examine the mathematical relations with the object of systematically reducing them to such a form as would give the utmost brevity and simplicity to the process of calculation, and, what is of still greater moment, obviate any possible risk of error. The resulting method which I then followed was afterwards approved and expressly adopted by the Committee of Actuaries in the computation of the "Tables exhibiting the Law of Mortality, deduced from the Combined Experience of Seventeen Life Assurance Offices;" and the superintendence of those calculations was eventually entrusted to the author, who was a member of that committee. In the present paper it is principally intended, amongst other matters relating to the same subject, to give a short exposition of this particular method, since but a cursory outline of it appeared in the pamphlet referred to; and, with a view to the practical utility of this object, it may be premised that the inquiry will admit of the most satisfactory and comprehensive application, if we suppose the investigation to appertain to the registered experience of a Life Office, since in this case the returns may be presumed to supply complete information with respect to each individual life.

For an accurate discussion of this kind the essential materials of information with respect to each life are—

1. The Office age on entry, or the date of birth;
2. The date of admission;
3. The date of exit, if such event has taken place;
4. The cause of such exit, whether by discontinuance or by death.

We shall, however, for the occasion of presenting an example in numbers, first consider the subject under ordinary conditions, in which, for the purpose of facility and simplicity, the particulars are not given with the exactitude here prescribed. Afterwards it will be easy to show what modification will be needed, if in any investigation the information were accurately defined and greater precision were required in the method of computation.

In the "Experience of Offices" only the calendar years, instead of the dates of entry and exit, were deemed to be sufficient for the purpose, it being considered that large numbers, promiscuously spread over any given year, when taken in combination, may on

the average, be practically assumed with tolerable correctness as uniformly dating from the middle of the year. For a like reason, only the current year of age, or the Office age, is required to be stated. Thus, by assuming the number of lives that entered in each current year of age to be, on the average, admitted also at the middle of such year a convenient coincidence will thereby subsist between calendar years and years of age; and the lives, one with another, may be considered as completing the Office age at the end of the calendar year of entry. The collection of the original data was therefore made in the following form :—

Containing the Experience of the Office up to the end of the Year 18 .									
For use of Office.	Current Age at Entry.	Year of		If by Death, D.		Sex, if Female, F.	Distinction into Town, T. Country, C. Irish, I.	Cause of Death.	Special Risks and Remarks.
		Entry.	Exit.						

When the list, according to this form, is made out and completed, the column containing the year of exit will at once disclose the policies that have ceased to exist through *discontinuance* or *death* during the period. The number of years they were severally in force, or the duration of each of these policies, will be deduced by taking the difference between the year of entry and the year of exit; and the current age of exit will be found by adding the years of duration to the current age at entry.

The policies which do not exhibit any year of exit are those which remained in force, and the lives of which were in *existence* at the end of the period of observation. In like manner, the years of duration of these policies will be found by taking the difference between the year of entry and the year to which the investigation extends, as stated at the top of the list. Also by adding, as before, the duration to the current age at entry we get, in each case, the current age to which the life attained and existed at the end of the period.

So far all the requisite details of the facts are separately known, and are ready for the purely mechanical work of classification and condensation. The chief objects to be accomplished will be to resolve the whole of the policies into three divisions—viz., 1, policies DISCONTINUED without death; 2, policies EXISTING at the termination of the period; and 3, policies cancelled by DEATH; which three divisions, we think, it will be found most convenient to give separately; from these to bring together all policies having the same age of entry; and, finally, to arrange and enumerate them, either with

respect to the years of duration or according to the ages attained. By this means the whole of the facts will become moulded into three tables, of which the following will serve as a specimen :—

TABLE (A).—*Enumeration of Discontinuances.*

Current Age of Entry	Year of Duration of Policy																			Total for Age of Entry
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	..
4	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	1
5	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	1
6	..	1	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	4
7	..	2	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	2
8	..	2	1	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	" .	5
9	..	2	..	2	3	..	..	4	..	1	..	1	..	..	..	..	..	..	" ..	13
10	..	4	3	4	1	1	..	2	..	2	..	..	1	..	..	..	..	..	" ..	18
11	..	4	2	2	..	3	..	1	1	1	..	..	..	..	..	..	..	..	" ..	14
12	..	3	2	4	1	2	..	4	..	..	1	..	..	..	..	..	..	..	" ..	17
13	..	11	5	1	1	..	..	2	1	1	..	..	..	..	..	..	..	..	" ..	22
14	..	4	3	3	2	1	2	3	2	1	..	2	..	..	..	..	..	..	" ..	23
15	..	8	5	2	3	3	..	5	..	..	1	..	..	..	..	..	..	..	" ..	27
16	1	6	6	2	3	1	1	2	..	1	..	..	..	..	..	..	..	..	" ..	23
17	..	10	6	4	4	2	1	8	..	..	1	..	..	..	..	..	..	..	" ..	36
18	..	8	3	12	10	4	3	1	..	..	..	..	..	..	..	..	..	..	" ..	41
19	..	15	17	21	5	1	..	8	..	2	..	..	..	..	..	..	..	..	" ..	69
20	1	38	59	9	6	8	2	4	..	..	..	..	..	..	..	..	..	..	" ..	127
&c.																				&c.

TABLE A.—*Enumeration of Discontinuances.*

Current Age of Entry	Current Age attained																			Total for Age of Entry
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	..
4	—	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	1
5	—	—	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	..	" ..	1
6	—	—	—	..	1	1	1	1	..	..	..	..	..	..	..	..	..	..	" ..	4
7	—	—	—	—	..	2	..	..	..	..	..	..	..	..	..	..	..	..	" ..	2
8	—	—	—	—	—	..	2	1	1	..	1	..	..	..	..	..	..	..	" ..	5
9	—	—	—	—	—	—	..	2	..	2	3	..	..	4	..	1	..	1	" ..	13
10	—	—	—	—	—	—	—	..	4	3	4	1	1	..	2	..	2	..	" ..	18
11	—	—	—	—	—	—	—	—	..	4	2	2	..	3	..	1	1	1	" ..	14
12	—	—	—	—	—	—	—	—	—	..	3	2	4	1	2	..	4	..	" ..	17
13	—	—	—	—	—	—	—	—	—	—	..	11	5	1	1	..	..	2	" ..	22
14	—	—	—	—	—	—	—	—	—	—	—	..	4	3	3	2	1	2	" ..	23
15	—	—	—	—	—	—	—	—	—	—	—	—	..	8	5	2	3	3	" .	27
16	—	—	—	—	—	—	—	—	—	—	—	—	—	1	6	6	2	3	" ..	23
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	10	6	4	" ..	36
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	8	3	" ..	41
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	15	" ..	69
20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	" ..	127
&c.																				&c.
(*r*)	..	..	..	..	3	3	3	4	5	9	13	16	14	21	19	22	27	35	&c.	8,188

TABLE (B).—*Enumeration of Policies finally Existing.*

Current Age of Entry	Year of Duration of Policy																		Total for Age of Entry
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	&c.	
3	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	,, ..	1
4	..	1	1		1	..	..	..	..	..	..	..	..	..	..	..	..	..	,, ..	3
5	..	..	..	3	1	..	..	..	..	..	..	..	..	..	..	..	..	..	,, ..	4
6	1	1	2	2	2	..	..	..	..	..	..	..	1	..	..	..	..	..	,, ..	9
7	1	3	1	3	1	2	1	..	..	..	1	2	1	..	..	..	..	..	,, ..	16
8	..	2	3	..	3	1	1	..	2	1	2	1	..	..	..	..	..	..	,, ..	16
9	5	1	2	4	1	3	1	2	3	1	..	1	2	..	..	..	..	..	,, ..	26
10	10	2	3	3	6	3	1	3	1		..	2	1	2	..	..	..	..	,, ..	37
11	2	1	..	3	4	..	3	3	1	3	.	..	2	2	1	..	..	..	,, ..	25
12	8	2	2	4	6	1	4	4	..	1	1	1	2	..	1	..	..	..	,, ..	37
13	8	4	4	4	6	4	..	2	2	1	1	3	2	..	..	..	..	..	,, ..	41
14	5	1	4	6	6	6	3	3	5	2	2	1	1	1	1	..	1	..	,, ..	48
15	8	7	4	3	6	1	2	6	..	1	..	..	5	.	..	..	1	..	,, ..	44
16	7	2	5	14	4	7	2	4	1	5	2	2	6	5	2	..	1	..	,, ..	69
17	8	16	10	14	6	5	3	..	2	1	..	1	3	2	1	..	..	1	,, ..	73
18	13	13	3	9	9	4	3	2	3	4	2	1	3	1	..	..	..	..	,, ..	70
19	18	9	8	8	4	7	1	..	..	5	2	2	1	1	5	..	..	..	,, ..	71
20	14	18	7	8	4	7	6	1	3	4	3	3	4	5	..	..	..	1	,, ..	88
&c.																				&c.

TABLE B.—*Enumeration of Policies finally Existing.*

Current Age of Entry	Current Age attained																			Total for Age of Entry
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
3	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	,, ..	1
4	—	..	1	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	,, ..	3
5	—	—	..	..	..	3	1	..	..	..	..	..	..	..	..	..	..	..	,, ..	4
6	—	—	—	1	1	2	2	2	..	..	..	..	..	..	..	1	..	..	,, ..	9
7	—	—	—	—	1	3	1	3	1	2	1	..	..	..	1	2	1	..	,, ..	16
8	—	—	—	—	—	..	2	3	..	3	1	1	..	2	1	2	1	.	,, ..	16
9	—	—	—	—	—	—	5	1	2	4	1	3	1	2	3	1	..	1	,, ..	26
10	—	—	—	—	—	—	—	10	2	3	3	6	3	1	3	1	..	..	,, ..	37
11	—	—	—	—	—	—	—	—	2	1	..	3	4	..	3	3	1	3	,, ..	25
12	—	—	—	—	—	—	—	—	—	8	2	2	4	6	1	4	4	..	,, ..	37
13	—	—	—	—	—	—	—	—	—	—	8	4	4	4	6	4	..	2	,, ..	41
14	—	—	—	—	—	—	—	—	—	—	—	5	1	4	6	6	6	3	,, ..	48
15	—	—	—	—	—	—	—	—	—	—	—	—	8	7	4	3	6	1	,, ..	44
16	—	—	—	—	—	—	—	—	—	—	—	—	—	7	2	5	14	4	,, ..	69
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	8	16	10	14	,, ..	73
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13	13	3	,, ..	70
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	18	9	,, ..	71
20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	14	,, ..	88
&c.																				&c.
(R)	..	..	1	2	2	10	11	19	7	21	16	24	25	33	38	61	74	54	&c.	20,710

TABLE (C).—*Enumeration of Deaths.*

Current Age of Entry	Year of Duration of Policy																		Total for Age of Entry
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
4	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
5	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
6	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
7	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	..	..	..	„ ..	1
8	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
9	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
10	..	..	..	1	..	1	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	2
11	..	..	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	3
12	..	..	..	..	1	..	..	1	..	..	..	..	..	..	..	..	..	..	„ ..	2
13	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
14	..	1	..	..	..	1	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	2
15	1	..	1	..	..	1	..	..	1	..	..	..	..	..	..	..	..	..	„ ..	4
16	..	1	..	..	1	..	..	..	..	1	..	..	..	..	..	..	..	..	„ ..	3
17	..	..	1	..	2	2	..	1	..	..	..	..	..	..	..	..	..	..	„ ..	6
18	..	3	1	1	1	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	6
19	1	..	1	..	..	..	..	..	..	..	1	..	..	..	..	..	..	..	„ ..	3
20	..	5	5	1	2	..	1	1	2	..	..	..	..	..	..	..	..	..	„ ..	17
&c.																				&c.

TABLE C.—*Enumeration of Deaths.*

Current Age of Entry	Current Age attained																		Total for Age of Entry
	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	&c.	
3	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
4	—	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
5	—	—	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
6	—	—	—	..	..	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
7	—	—	—	—	..	..	..	..	..	..	..	1	..	..	..	..	..	..	„ ..	1
8	—	—	—	—	—	..	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
9	—	—	—	—	—	—	..	..	..	..	..	..	..	..	..	..	..	..	„ ..	..
10	—	—	—	—	—	—	—	..	..	..	1	..	1	..	..	..	..	..	„ ..	2
11	—	—	—	—	—	—	—	—	..	..	1	1	1	..	..	..	..	..	„ ..	3
12	—	—	—	—	—	—	—	—	—	..	..	..	..	1	..	..	1	..	„ ..	2
13	—	—	—	—	—	—	—	—	—	—	..	..	..	..	..	..	..	..	„ ..	..
14	—	—	—	—	—	—	—	—	—	—	—	..	1	..	..	..	1	..	„ ..	2
15	—	—	—	—	—	—	—	—	—	—	—	—	1	..	1	..	..	1	„ ..	4
16	—	—	—	—	—	—	—	—	—	—	—	—	—	..	1	..	..	1	„ ..	3
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	..	1	..	„ ..	6
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	3	1	„ ..	6
19	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	..	„ ..	3
20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	..	„ ..	17
&c.																				&c.
(D)	..	..	..	..	..	..	..	..	..	..	2	2	4	1	2	..	7	3	&c.	2,482

The numbers here tabulated separately in the manner prescribed correctly represent, as far as they go, a combination of the four principal "Experience Tables," viz.:—

A(1), Male lives—Town, comprising .	.	16,097 lives
A(2), Male lives—Country ,,	.	11,926 ,,
A(4), Female lives—Town ,,	.	1,448 ,,
A(5), Female lives—Country ,,	.	1,909 ,,
Total . .	.	31,380 lives

The upper set of tables, (A), (B), (C), in which the numbers are distributed according to the years of duration of the policies, will have as many columns as the number of years over which the observations extend. When this period embraces only a limited number of years, the same being less than the number of ages, it will then be the most compact form in which to present the experience of an Office.

The lower set of tables, A, B, C, will have as many columns as may be sufficient to include all ages, and the contents will evidently consist of the same numbers as the former set; the only difference being, that the numbers which occupy the several horizontal lines are successively translated a place further to the right.

It will not be necessary in any case to prepare both of these sets of tables, as the data for any subsequent calculation can be readily extracted from either of them as they stand. The numbers we shall have occasion to combine will be those which appertain to the same acquired ages, and will, in the upper tables, ascend diagonally from the numbers immediately contiguous to each age of entry on the left; whereas in the more elongated tables, A, B, C, they will stand somewhat more conveniently in vertical columns. From sets of tables, in either of these forms, any number of which, representing different experiences, might be incorporated by simple addition, the various questions which relate to the value of selection of assured lives may be determined. If required, a table of mortality might be constructed for each separate age of entry, or for any distinct groups of ages; but it is needless to add, that such inquiries would be futile with small numbers.

The totals contained in the last column of each table are the sums of the numbers entered in each horizontal line when the table is completed. These numbers, collected from the three tables, will give the total number that entered in each current year of age as follows:—

Number that entered in each Age.

Current Age of Entry	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Total No. Disc.	..	1	1	4	2	5	13	18	14	17	22	23	27	23	36	41	69	127
„ „ Existing	1	3	4	9	16	16	26	37	25	37	41	48	44	69	73	70	71	88
„ „ Deaths........	..	..	..	..	1	..	..	2	3	2	..	2	4	3	6	6	3	17
Total No. Issued (n)	1	4	5	13	19	21	39	57	42	56	63	73	75	95	115	117	143	232

For the computation of a table of mortality from the entire experience, which is, indeed, the main object in view, only the totals of the tables A, B, C will be needed; and if such a table of mortality should be the sole subject of inquiry, these totals might otherwise be most expeditiously found from the original returns* by filling in the ages of exit in the blank column and then simply counting the number of policies which answer to each of the several ages of entry; and afterwards counting, in like manner, the number of policies which answer to each age of exit, only remembering to distinguish these latter in the three divisions, so as to obtain separately the numbers (r), (R), and (D). For the purpose of following out the example already introduced, a complete statement of these totals is here tabulated.†

* *See* the Form on page 77.

† An increased facility has recently been given to the processes of analysing the facts of a mortality experience by simply conveying the original particulars, with respect to each individual life, upon a separate card, specially designed to receive them. By this means the object of classifying each particular element, whatever it may be, is accomplished by the purely mechanical operation of shuffling the cards into the required order, without incurring the necessity, in any case, of making preliminary extracts of the several details. This is, undoubtedly, the most prominent advantage obtained, as there is always a much greater liability to error in making long and tedious transcriptions than in performing the operations of calculation. The plan, here briefly described, was first made publicly known in "An Account of the Processes employed in getting out the Mortality Experience of the Economic Life Assurance Society," and was the suggestion of Mr. O. G. Downes, of that Office. An appropriate and well-considered form of card, calculated to meet every requirement and convenience, is presented in the pamphlet referred to, and the method is appropriately designated the "card system." It is also pointed out that "it affords the means of making any deductions required, as it is only necessary to throw the cards into hotch-potch, and then rearrange them in the order needed to accomplish the required object—a process which would afford pleasant fireside amusement to any domesticated actuary and his family." The only disadvantage that could attend the method would appear to be the possibility of some of the cards, in their dignified isolation and independence, getting accidentally astray from the general mass, and carrying with them their separate contents, a casualty which could not happen when the materials are permanently entered on sheets, prepared for their reception. The mere idea of this supposed objection will, at least, induce a due amount of caution in those who, in the discussion of any branch of statistics, may be disposed to avail themselves of the manifold advantages of the card system.

TABLE D.—*Total Enumerations in each Year of Age.*

Current Year of Age	Entered (n)	Discontinued (r)	Existing at the Termination (R)	Died (D)
3	1	..	..	..
4	4	..	..	..
5	5	..	1	..
6	13	..	2	..
7	19	3	2	..
8	21	3	10	..
9	39	3	11	..
10	57	4	19	..
11	42	5	7	..
12	56	9	21	..
13	63	13	16	2
14	73	16	24	2
15	75	14	25	4
16	95	21	33	1
17	115	19	38	2
18	117	22	61	..
19	143	27	74	7
20	232	35	54	3
21	286	83	78	11
22	389	219	92	13
23	453	102	115	6
24	509	119	131	11
25	722	184	183	10
26	710	158	181	14
27	878	188	266	16
28	910	212	286	28
29	1,000	219	332	24
30	1,083	210	339	22
31	1,059	243	379	31
32	1,073	248	428	41
33	1,123	255	489	27
34	1,175	269	555	57
35	1,145	271	591	51
36	1,142	266	639	67
37	1,089	285	616	49
38	1,027	266	732	67
39	967	271	667	70
40	1,055	264	709	43
41	904	272	711	71
42	901	266	679	68
43	887	235	683	54
44	849	234	693	71
45	801	201	636	62
	23,307	5,734	11,608	1,005
	23,307	5,734	11,608	1,005
46	667	204	694	70
47	703	206	695	54
48	640	178	591	66
49	657	182	569	79
50	593	152	536	57
51	489	166	513	57
52	522	168	495	56
53	448	134	486	59
54	368	134	442	64
55	373	144	421	61
56	359	93	387	76
57	329	88	358	64
58	302	89	324	48
59	267	79	277	65
60	245	72	288	56
61	183	57	312	55
62	178	43	276	49
63	159	57	212	37
64	131	38	196	45
65	119	22	169	61
66	85	35	127	39
67	79	25	148	35
68	47	32	99	33
69	28	15	89	28
70	26	9	84	25
71	20	7	98	27
72	14	4	40	15
73	14	4	48	14
74	11	2	32	15
75	5	7	23	11
76	3	2	16	19
77	3	1	16	6
78	3	2	12	10
79	2	..	16	11
80	..	1	6	3
81	..	1	1	2
82	..	..	..	..
83	..	..	..	..
84	1	..	3	1
85	..	1	2	3
86	..	..	..	1
87	..	..	..	..
88	..	..	1	..
Total	31,380	8,188	20,710	2,482

The data thus prepared, on which the calculation of the required table of mortality is to be founded, embody a complete analysis and classification of a mass of observations which extends over a given number of years. To proceed with the calculation of the the rates of mortality the principal requirement will be to ascertain how many lives, during that period, have entered and passed through each year of age, so as to be enabled to compare the same with the number of deaths that have respectively occurred in the same years. It will also appear that these estimates will not be affected by any chronological considerations, since a question as to when any specified life or lives entered the year of age is quite immaterial and forms no part of the inquiry. We only require to know how many lives have passed through a proposed year of age at all the various times during the period of observation, but without any reference to those times.

In any proposed current year of age let

N denote the number of lives in existence at the commencement of the period;

$$\left.\begin{matrix} n \\ r \\ D \end{matrix}\right\} \text{the number of} \left\{\begin{matrix} \text{admissions} \\ \text{retirements} \\ \text{deaths} \end{matrix}\right\} \text{in the course of the period;}$$

R the number remaining in existence on the list at the termination of the period;

ε the total number that, in the course of the period, have passed the point of commencement of the proposed year of age, or the total number that have entered as survivors at the commencement; and

$N_1, n_1, r_1, D_1, R_1, \varepsilon_1$, the like numbers for the next higher year of age, &c.

Then it is evident that the number of lives which have existed at the end of the year of age may be directly deduced from the number at the commencement, by merely taking into account the several operations or events that have occurred in the same year. Thus, at the end of the proposed year of age, the survivors ε, at the beginning of that year, will, in the course of the year of age, become augmented by $N+n$ and diminished by $R+r+D$. The resulting number that existed at the end of the year is therefore $\varepsilon+(N+n)-(R+r+D)$, and this must necessarily be the number ε_1 that enter by survivorship the next higher year of age. Whence, if

$$\left.\begin{aligned} \theta &= (N+n)-r \\ \omega &= (N+n)-(R+r) \\ &= \theta - R \end{aligned}\right\} \quad . \quad . \quad . \quad (1),$$

then will $$\epsilon_1 = \epsilon + \omega - \mathrm{D},$$

$$\left.\begin{aligned} \text{or } \Delta\epsilon &= \omega - \mathrm{D} \\ \epsilon_1 &= \epsilon + \Delta\epsilon \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This simple and obvious deduction being applied consecutively to each retrogressive year of age will put us in possession of the subsisting relations which determine and guide the calculation.

Thus, by commencing at any age x, the several relations put down in a descending order are

$$\begin{aligned} \epsilon_x &= \epsilon_{x-1} + \omega_{x-1} - \mathrm{D}_{x-1} \\ &= \epsilon_{x-1} + (\omega - \mathrm{D})_{x-1} \\ \epsilon_{x-1} &= \epsilon_{x-2} + (\omega - \mathrm{D})_{x-2} \\ \epsilon_{x-2} &= \epsilon_{x-3} + (\omega - \mathrm{D})_{x-3} \\ \&\text{c.} & \qquad \&\text{c.} \end{aligned}$$

Hence, by successive substitution, or by adding the equations together and cancelling identities, which leads to the same result, we obtain

$$\begin{aligned} \epsilon_x &= (\omega - \mathrm{D})_{x-1} + (\omega - \mathrm{D})_{x-2} + (\omega - \mathrm{D})_{x-3} + \&\text{c.} \\ &= \Sigma_{x-1}(\omega - \mathrm{D}) = \Sigma_{x-1}(\Delta\epsilon) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3), \end{aligned}$$

in which it will be seen that the characteristic Σ, as indicated, includes the accumulation of values for all the years of age which precede the year under consideration. And thus we find, by continued summation, the total number of lives that from time to time, in the course of the period, have existed as survivors at the commencement of each successive year of age.

Now, according to the blank form specified on page 77, in which the original returns were entered, the several numbers representing the experience are inclusive of the whole duration of the initial and terminal years of observation; and it will thence follow that the number R, finally in existence, must be regarded as appertaining to the end of the corresponding year of age. But the numbers N, n, r, which compose θ, or, in other words, the several entrances and exits which, in the year of age, have operated in disturbing the number of lives otherwise than by death are distributed through the year of age at indiscriminate intervals and may, with large numbers, be practically considered, on the average, as having reference to the middle of the year, or as having undergone half the chance of death in one year.* The RISK may therefore be regarded as that of $\epsilon + \frac{1}{2}\theta$

* The average interval by which the date of entry precedes the birthday is here assumed to be half a year, which may be expected to be somewhat in excess, since in some cases it is natural to surmise a disposition to expedite an assurance when a coming birthday, in near perspective, is suggestive of a higher rate of premium. According to actual experience the average interval is found to be about three-eighths of a year. The

persons entering upon the year, and the uninterrupted survivors of the same continuing exposed to its termination. Therefore, as D is the number of deaths that have occurred in the same year of age, the corresponding number of survivors may be considered to be $\varepsilon+\frac{1}{2}\theta-\mathrm{D}$. Hence the proportionate rate of mortality in the year is

$$m=\frac{\mathrm{D}}{\varepsilon+\frac{1}{2}\theta};$$

and the proportion that survive, or the probability of surviving the year is

$$p=\frac{\varepsilon+\frac{1}{2}\theta-\mathrm{D}}{\varepsilon+\frac{1}{2}\theta}.$$

To abbreviate, let

$$\left.\begin{aligned} \beta&=\varepsilon+\tfrac{1}{2}\theta \\ \gamma&=\beta-\mathrm{D} \\ \text{then } m&=\frac{\mathrm{D}}{\beta} \\ p&=\frac{\gamma}{\beta} \end{aligned}\right\} \quad \dots \quad (4).$$

Also if, according to the usual notation, l_x, l_{x+1}, &c., denote the numbers living in the required table of mortality at the ages x, $x+1$, &c., the logarithms of these numbers for all ages in the table are hence deduced from the logarithms of p, by successive addition, beginning with a suitable number as an arbitrary radix for the first age of the table. Thus,

$$\begin{aligned} \log l_{x+1}&=\log p_x \quad +\log l_x \\ \log l_{x+2}&=\log p_{x+1}+\log l_{x+1} \\ \&\text{c.} & \qquad\quad \&\text{c.} \end{aligned}$$

The following example, worked out at length, will perhaps best exemplify the practical application of the preceding formulæ. The data are taken from Table D, observing that $\mathrm{N}=0$ for all ages, since the returns record every life from the year of entry.

As the numbers for the younger ages are small the mortality table is made to begin with age 10, and the radix is $l_{10}=100000$.

For the latter ages some of the numbers come out negative, and it then becomes requisite to attend to the algebraic signs.

The values of ε are completed horizontally by successive addition, and the final vanishing of these numbers after the last age of the table will, up to this point, supply a check on the accuracy of the calculation.

difference is however not material. As the lives are by this means accounted nearly one-eighth of a year younger than they really are, the ultimate tendency will be to slightly increase the rate of mortality. A more accurate method of computation is given further on.

The values of $\log l$ are also completed horizontally in a similar manner.

In these calculations it will be observed that the annual rate of mortality, m, is given under its usual acceptation, viz.:—

$$m_x = \frac{l_x - l_{x+1}}{l_x} = \frac{\text{decrement}}{\text{number living}}.$$

Completed Age	2	3	4	5	6	7	8	9
$N+n$	1	4	5	13	19	21	39	57
r	..	..	..	..	3	3	3	4
$(N+n)-r=\theta$	1	4	5	13	16	18	36	53
R	..	..	1	2	2	10	11	19
$\theta-R=\omega$	1	4	4	11	14	8	25	34
D	..	..	..	..	..	..	..	..
$\omega-D=\Delta\varepsilon$	1	4	4	11	14	8	25	34
ε	0	1	5	9	20	34	42	67
$\frac{1}{2}\theta$	0·5	2·0	2·5	6·5	8·0	9·0	18·0	26·5
$\varepsilon+\frac{1}{2}\theta=\beta$	0·5	3·0	7·5	15·5	28·0	43·0	60·0	93·5
$\beta-D=\gamma$	0·5	3·0	7·5	15·5	28·0	43·0	60·0	93·5

Age	10	11	12	13	14
$N+n$	42	56	63	73	75
r	5	9	13	16	14
θ	37	47	50	57	61
R	7	21	16	24	25
ω	30	26	34	33	36
D	..	..	2	2	4
$\Delta\varepsilon$	30	26	32	31	32
ε	101	131	157	189	220
$\frac{1}{2}\theta$	18·5	23·5	25·0	28·5	30·5
β	119·5	154·5	182·0	217·5	250·5
γ	119·5	154·5	180·0	215·5	246·5
$\log\beta$	2·07737	2·18893	2·26007	2·33746	2·39881
$\log\gamma$	2·07737	2·18893	2·25527	2·33345	2·39182
$\log p$	0·00000	0·00000	9·99520	9·99599	9·99301
$\log l$	5·00000	5·00000	5·00000	4·99520	4·99119
l	100000	100000	100000	98901	97992
p	1·00000	1·00000	·98901	·99081	·98403
m	·00000	·00000	·01099	·00919	·01597

Age	15	16	17	18	19
N + n r	95 21	115 19	117 22	143 27	232 35
θ R	74 33	96 38	95 61	116 74	197 54
ω D	41 1	58 2	34 0	42 7	143 3
Δε ε ½θ	40 252 37·0	56 292 48·0	34 348 47·5	35 382 58·0	140 417 98·5
β γ	289·0 288·0	340·0 338·0	395·5 395·5	440·0 433·0	515·5 512·5
log β log γ	2·46090 2·45939	2·53148 2·52892	2·59715 2·59715	2·64345 2·63649	2·71223 2·70969
log p log l	9·99849 4·98420	9·99744 4·98269	0·00000 4·98013	9·99304 4·98013	9·99746 4·97317
l p m	96427 ·99653 ·00347	96093 ·99412 ·00588	95528 1·00000 ·00000	95528 ·98410 ·01590	94009 ·99417 ·00583

Age	20	21	22	23	24
N + n r	286 83	389 219	453 102	509 119	722 184
θ R	203 78	170 92	351 115	390 131	538 183
ω D	125 11	78 13	236 6	259 11	355 10
Δε ε ½θ	114 557 101·5	65 671 85·0	230 736 175·5	248 966 195·0	345 1214 269·0
β γ	658·5 647·5	756·0 743·0	911·5 905·5	1161·0 1150·0	1483·0 1473·0
log β log γ	2·81856 2·81124	2·87852 2·87099	2·95976 2·95689	3·06483 3·06070	3·17114 3·16820
log p log l	9·99268 4·97063	9·99247 4·96331	9·99713 4·95578	9·99587 4·95291	9·99706 4·94878
l p m	93461 ·98329 ·01671	91899 ·98281 ·01719	90319 ·99341 ·00659	89724 ·99054 ·00946	88875 ·99325 ·00675

Age	25	26	27	28	29
N + n	710	878	910	1000	1083
r	158	188	212	219	210
θ	552	690	698	781	873
R	181	266	286	332	339
ω	371	424	412	449	534
D	14	16	28	24	22
$\Delta\epsilon$	357	408	384	425	512
ϵ	1559	1916	2324	2708	3133
$\frac{1}{2}\theta$	276·0	345·0	349·0	390·5	436·5
β	1835·0	2261·0	2673·0	3098·5	3569·5
γ	1821·0	2245·0	2645·0	3074·5	3547·5
$\log\beta$	3·26364	3·35430	3·42700	3·49115	3·55261
$\log\gamma$	3·26031	3·35122	3·42243	3·48777	3·54992
$\log p$	9·99667	9·99692	9·99543	9·99662	9·99731
$\log l$	4·94584	4·94251	4·93943	4·93486	4·93148
l	88275	87601	86982	86072	85404
p	·99236	·99293	·98953	·99225	·99383
m	·00764	·00707	·01047	·00775	·00617

Age	30	31	32	33	34
N + n	1059	1073	1123	1175	1145
r	243	248	255	269	271
θ	816	825	868	906	874
R	379	428	489	555	591
ω	437	397	379	351	283
D	31	41	27	57	51
$\Delta\epsilon$	406	356	352	294	232
ϵ	3645	4051	4407	4759	5053
$\frac{1}{2}\theta$	408·0	412·5	434·0	453·0	437·0
β	4053·0	4463·5	4841·0	5212·0	5490·0
γ	4022·0	4422·5	4814·0	5155·0	5439·0
$\log\beta$	3·60778	3·64968	3·68494	3·71700	3·73957
$\log\gamma$	3·60444	3·64567	3·68251	3·71223	3·73552
$\log p$	9·99666	9·99599	9·99757	9·99523	9·99595
$\log l$	4·92879	4·92545	4·92144	4·91901	4·91424
l	84877	84227	83453	82987	82080
p	·99234	·99081	·99442	·98908	·99072
m	·00766	·00919	·00558	·01092	·00928

Age	35	36	37	38	39
N+n	1142	1089	1027	967	1055
r	266	285	266	271	264
θ	876	804	761	696	791
R	639	616	732	667	709
ω	237	188	29	29	82
D	67	49	67	70	43
$\Delta\epsilon$	+170	+139	−38	−41	+39
ϵ	5285	5455	5594	5556	5515
$\frac{1}{2}\theta$	438·0	402·0	380·5	348·0	395·5
β	5723·0	5857·0	5974·5	5904·0	5910·5
γ	5656·0	5808·0	5907·5	5834·0	5867·5
$\log\beta$	3·75762	3·76768	3·77630	3·77115	3·77162
$\log\gamma$	3·75251	3·76403	3·77140	3·76597	3·76845
$\log p$	9·99489	9·99635	9·99510	9·99482	9·99683
$\log l$	4·91019	4·90508	4·90143	4·89653	4·89135
l	81319	80367	79695	78801	77866
p	·98830	·99163	·98878	·98814	·99273
m	·01170	·00837	·01122	·01186	·00727

Age	40	41	42	43	44
N+n	904	901	887	849	801
r	272	266	235	234	201
θ	632	635	652	615	600
R	711	679	683	693	636
ω	−79	−44	−31	−78	−36
D	71	68	54	71	62
$\Delta\epsilon$	−150	−112	−85	−149	−98
ϵ	5554	5404	5292	5207	5058
$\frac{1}{2}\theta$	316·0	317·5	326·0	307·5	300·0
β	5870·0	5721·5	5618·0	5514·5	5358·0
γ	5799·0	5653·5	5564·0	5443·5	5296·0
$\log\beta$	3·76864	3·75751	3·74958	3·74151	3·72900
$\log\gamma$	3·76335	3·75232	3·74539	3·73588	3·72395
$\log p$	9·99471	9·99481	9·99581	9·99437	9·99495
$\log l$	4·88818	4·88289	4·87770	4·87351	4·86788
l	77300	76364	75457	74733	73770
p	·98789	·98812	·99040	·98712	·98844
m	·01211	·01188	·00960	·01288	·01156

Age	45	46	47	48	49
N+n	667	703	640	657	593
r	204	206	178	182	152
θ	463	497	462	475	441
R	694	695	591	569	536
ω	−231	−198	−129	−94	−95
D	70	54	66	79	57
$\Delta\epsilon$	−301	−252	−195	−173	−152
ϵ	4960	4659	4407	4212	4039
$\frac{1}{2}\theta$	231·5	248·5	231·0	237·5	220·5
β	5191·5	4907·5	4638·0	4449·5	4259·5
γ	5121·5	4853·5	4572·0	4370·5	4202·5
log β	3·71529	3·69086	3·66633	3·64831	3·62936
log γ	3·70940	3·68605	3·66011	3·64053	3·62351
log p	9·99411	9·99519	9·99378	9·99222	9·99415
log l	4·86283	4·85694	4·85213	4·84591	4·83813
l	72917	71935	71143	70131	68886
p	·98653	·98899	·98578	·98225	·98662
m	·01347	·01101	·01422	·01775	·01338

Age	50	51	52	53	54
N+n	489	522	448	368	373
r	166	168	134	134	144
θ	323	354	314	234	229
R	513	495	486	442	421
ω	−190	−141	−172	−208	−192
D	57	56	59	64	61
$\Delta\epsilon$	−247	−197	−231	−272	−253
ϵ	3887	3640	3443	3212	2940
$\frac{1}{2}\theta$	161·5	177·0	157·0	117·0	114·5
β	4048·5	3817·0	3600·0	3329·0	3054·5
γ	3991·5	3761·0	3541·0	3265·0	2993·5
log β	3·60729	3·58172	3·55630	3·52231	3·48494
log γ	3·60114	3·57530	3·54913	3·51388	3·47618
log p	9·99385	9·99358	9·99283	9·99157	9·99124
log l	4·83228	4·82613	4·81971	4·81254	4·80411
l	67964	67009	66025	64944	63696
p	·98594	·98533	·98363	·98078	·98003
m	·01406	·01467	·01637	·01922	·01997

Age	55	56	57	58	59
N + n	359	329	302	267	245
r	93	88	89	79	72
θ	266	241	213	188	173
R	387	358	324	277	288
ω	−121	−117	−111	−89	−115
D	76	64	48	65	56
$\Delta\epsilon$	−197	−181	−159	−154	−171
ϵ	2687	2490	2309	2150	1996
$\frac{1}{2}\theta$	133·0	120·5	106·5	94·0	86·5
β	2820·0	2610·5	2415·5	2244·0	2082·5
γ	2744·0	2546·5	2367·5	2179·0	2026·5
$\log\beta$	3·45025	3·41672	3·38301	3·35102	3·31859
$\log\gamma$	3·43838	3·40594	3·37429	3·33826	3·30675
$\log p$	9·98813	9·98922	9·99128	9·98724	9·98816
$\log l$	4·79535	4·78348	4·77270	4·76398	4·75122
l	62424	60741	59252	58074	56392
p	·97304	·97548	·98012	·97105	·97311
m	·02696	·02452	·01988	·02895	·02689

Age	60	61	62	63	64
N + n	183	178	159	131	119
r	57	43	57	38	22
θ	126	135	102	93	97
R	312	276	212	196	169
ω	−186	−141	−110	−103	−72
D	55	49	37	45	61
$\Delta\epsilon$	−241	−190	−147	−148	−133
ϵ	1825	1584	1394	1247	1099
$\frac{1}{2}\theta$	63·0	67·5	51·0	46·5	48·5
β	1888·0	1651·5	1445·0	1293·5	1147·5
γ	1833·0	1602·5	1408·0	1248·5	1086·5
$\log\beta$	3·27600	3·21788	3·15987	3·11177	3·05975
$\log\gamma$	3·26316	3·20480	3·14860	3·09639	3·03603
$\log p$	9·98716	9·98692	9·98873	9·98462	9·97628
$\log l$	4·73938	4·72654	4·71346	4·70219	4·68681
l	54876	53277	51696	50372	48619
p	·97087	·97033	·97438	·96521	·94685
m	·02913	·02967	·02562	·03479	·05315

Age	65	66	67	68	69
N + n r	85 35	79 25	47 32	28 15	26 9
θ R	50 127	54 148	15 99	13 89	17 84
ω D	−77 39	−94 35	−84 33	−76 28	−67 25
$\Delta\epsilon$ ϵ $\frac{1}{2}\theta$	−116 966 25·0	−129 850 27·0	−117 721 7·5	−104 604 6·5	−92 500 8·5
β γ	991·0 952·0	877·0 842·0	728·5 695·5	610·5 582·5	508·5 483·5
$\log\beta$ $\log\gamma$	2·99607 2·97864	2·94300 2·92531	2·86243 2·84230	2·78569 2·76530	2·70629 2·68440
$\log p$ $\log l$	9·98257 4·66309	9·98231 4·64566	9·97987 4·62797	9·97961 4·60784	9·97811 4·58745
l	46035	44224	42459	40536	38677
p m	·96066 ·03934	·96009 ·03991	·95471 ·04529	·95414 ·04586	·95085 ·04915

Age	70	71	72	73	74
N + n r	20 7	14 4	14 4	11 2	5 7
θ R	13 98	10 40	10 48	9 32	−2 23
ω D	−85 27	−30 15	−38 14	−23 15	−25 11
$\Delta\epsilon$ ϵ $\frac{1}{2}\theta$	−112 408 6·5	−45 296 5·0	−52 251 5·0	−38 199 4·5	−36 161 −1·0
β γ	414·5 387·5	301·0 286·0	256·0 242·0	203·5 188·5	160·0 149·0
$\log\beta$ $\log\gamma$	2·61752 2·58827	2·47857 2·45637	2·40824 2·38382	2·30856 2·27531	2·20412 2·17319
$\log p$ $\log l$	9·97075 4·56556	9·97780 4·53631	9·97558 4·51411	9·96675 4·48969	9·96907 4·45644
l	36776	34380	32667	30881	28605
p m	·93487 ·06513	·95017 ·04983	·94532 ·05468	·92630 ·07370	·93126 ·06874

Age	75	76	77	78	79
N + n	3	3	3	2	..
r	2	1	2	..	1
θ	1	2	1	2	−1
R	16	16	12	16	6
ω	−15	−14	−11	−14	−7
D	19	6	10	11	3
$\Delta\epsilon$	−34	−20	−21	−25	−10
ϵ	125	91	71	50	25
$\frac{1}{2}\theta$	0·5	1·0	0·5	1·0	−0·5
β	125·5	92·0	71·5	51·0	24·5
γ	106·5	86·0	61·5	40·0	21·5
$\log \beta$	2·09864	1·96379	1·85431	1·70757	1·38917
$\log \gamma$	2·02735	1·93450	1·78888	1·60206	1·33244
$\log p$	9·92871	9·97071	9·93457	9·89449	9·94327
$\log l$	4·42551	4·35422	4·32493	4·25950	4·15399
l	26639	22606	21131	18176	14256
p	·84861	·93478	·86014	·78431	·87755
m	·15139	·06522	·13986	·21569	·12245

Age	80	81	82	83	84
N + n	..	..	..	1	..
r	1	..	..	..	1
θ	−1	..	..	+1	−1
R	1	..	..	3	2
ω	−2	..	..	−2	−3
D	2	..	..	1	3
$\Delta\epsilon$	−4	..	..	−3	−6
ϵ	15	11	11	11	8
$\frac{1}{2}\theta$	−0·5	..	..	+0·5	−0·5
β	14·5	11·0	11·0	11·5	7·5
γ	12·5	11·0	11·0	10·5	4·5
$\log \beta$	1·16137	1·04139	1·04139	1·06070	0·87506
$\log \gamma$	1·09691	1·04139	1·04139	1·02119	0·65321
$\log p$	9·93554	0·00000	0·00000	9·96049	9·77815
$\log l$	4·09726	4·03280	4·03280	4·03280	3·99329
l	12510	10784	10784	10784	9847
p	·86206	1·00000	1·00000	·91304	·60000
m	·13794	·00000	·00000	·08696	·40000

Age	85	86	87	88
$N+n$ r	.. ..	.. ..	.. ..	.. ..
θ R	.. ..	.. ..	.. 1	.. ..
ω D	.. 1	.. ..	−1 ..	.. ..
$\Delta\epsilon$ ϵ $\frac{1}{2}\theta$	−1 2 ..	.. 1 ..	−1 1 ..	.. .. ..
β γ	2·0 1·0	1·0 1·0	1·0 1·0	.. ..
$\log\beta$ $\log\gamma$	0·30103 0·00000	0·00000 0·00000	0·00000 0·00000	.. ..
$\log p$ $\log l$	9·69897 3·77144	0·00000 3·47041	0·00000 3·47041	.. ..
l p m	5908 ·50000 ·50000	2954 1·00000 ·00000	2954 1·00000 ·00000	.. .. ..

The computed values of the number living and of the rate of mortality per cent., for every year of age, are laid down in two drawings, so as to exhibit at one view the general character and consistency of the results. In the latter diagram is also shown the corresponding curve, according to the adjusted Experience Table. It will be found on examination that the irregularities that stand out so conspicuously, at the earliest and latest ages, may be traced to the smallness of the numbers (β) that enter into the calulation. The results would have been much more valuable if the original experience had extended over a longer period.

If the number living be taken as the element for final adjustment according to the rules given in a former paper (vol. xii., pages 140–1), the same aggregate tabular mortality or decrement must necessarily be retained between all points of actual coincidence, in whatever way the intermediate numbers may be modified; and this is certainly a legitimate and desirable condition. The minutiæ of accurate distribution, however important, ought to be a secondary consideration to that of a true embodiment of the total mortality. An expert method of effecting a tabular distribution of the rate of mortality, by a special application of Gompertz's formula,

is given by Mr. Jellicoe in his valuable paper inserted in vol. ii., page 199, to which we shall have occasion to refer hereafter.

In the investigation of the experience of the Amicable Society, by the late Mr. Galloway, the ages on admission are assumed, as in the foregoing, to be half a year less than the Office ages; but in other respects the data are collected according to a succession of years, reckoned, in each case, from the exact date of entry as an epoch. The method of working these up will be à little different. Thus if we here employ notation, analogous to that which has preceded, the original numbers to be used in the computation are

$$\begin{aligned} n' &= \text{number of admissions at age } x-\tfrac{1}{2}; \\ \left.\begin{aligned} R' &= \text{number finally existing} \\ r' &= \text{number discontinued} \\ D' &= \text{number of deaths} \end{aligned}\right\} & \begin{array}{l}\text{between the ages} \\ x-\tfrac{1}{2} \text{ and } x+\tfrac{1}{2}.\end{array} \end{aligned}$$

By taking the arithmetical mean values between each of these last numbers and the corresponding number for the preceding age, we shall then have the whole of the data, estimated in the way we have prescribed, viz.:—

$$\left.\begin{aligned} n &= n' && = \text{number of admissions} \\ R &= \tfrac{1}{2}(R'+R'_{-1}) && = \text{number finally existing} \\ r &= \tfrac{1}{2}(r'+r'_{-1}) && = \text{number discontinued} \\ D &= \tfrac{1}{2}(D'+D'_{-1}) && = \text{number of deaths} \end{aligned}\right\} \begin{array}{c}\text{in the } x\text{th year} \\ \text{of age.}\end{array}$$

Or, doubling to avoid fractions,

$$n=2n', \quad R=R'+R'_{-1}, \quad r=r'+r'_{-1}, \quad D=D'+D'_{-1}.$$

Hence, by taking these values, we can readily determine ϵ as before; but the number at risk will be $\epsilon+\frac{1}{2}\omega$, instead of $\epsilon+\frac{1}{2}\theta$, since the R lives finally existing are here assumed to be equally distributed over the year of age.

The implied hypothesis that lives admitted at ages of the form $x-\frac{1}{2}$ have their discontinuances equally distributed in years of the form $x-\frac{1}{2}$ to $x+\frac{1}{2}$ should be considered in the application of this method.

We shall now resume the original subject, with a more strict attention to minute details, and suppose the returns to supply the exact dates of birth, entry and exit. From these the precise ages at entry and exit can be stated to a fraction of a year; but the precise ages at death will not be required. The notation will be unchanged and the computation of the numbers ϵ, that enter the several years of age, will in every respect be the same. The modification to be made in the calculation will alone consist in the allow-

ances to be made for the several entrances and exits which occur in the year of age, and which, being individually known, we are no longer constrained to take at an assumed average value. These entrances and exits when combined determine the number ω, viz.,

$$\omega=(N+n)-(R+r);$$

and it is now required to find the exact portions of the year of age that are occupied by the constituent numbers, N, n, R, r.

First take the N lives that exist at the commencement, in the current year of age x, and let the exact ages of these lives be severally $x-\delta N_1$, $x-\delta N_2$, $x-\delta N_3$, &c., and let the sum of the various fractions of years, δN_1, δN_2, δN_3, &c., be denoted by $\Sigma\delta N$.

Similarly, let the exact ages of the n lives admitted in the year of age be severally $x-\delta n_1$, $x-\delta n_2$, $x-\delta n_3$, &c., and let the sum of the fractions δn_1, δn_2, δn_3, &c., be $\Sigma\delta n$.

Again, let the exact ages of the R lives finally existing, in the year of age, be severally $x-\delta R_1$, $x-\delta R_2$, $x-\delta R_3$, &c., and let the sum of the fractions δR_1, δR_2, δR_3, &c., be $\Sigma\delta R$.

And, lastly, let the exact ages of the r discontinuances be $x-\delta r_1$, $x-\delta r_2$, $x-\delta r_3$, &c., and let the sum of the fractions δr_1, δr_2, δr_3, &c., be $\Sigma\delta r$.

Then the more accurate number of lives at risk will evidently be

$$\beta=\varepsilon+(\Sigma\delta N+\Sigma\delta n)-(\Sigma\delta R+\Sigma\delta r).$$

The values of each set of fractions (δN, δn, δR, δr), may be put down either in decimals of a year or in days. In the latter case the sums would have to be divided by 365. The other steps of the calculation will be precisely the same as before.

In Table D the respective sums $\Sigma\delta N$, $\Sigma\delta n$, $\Sigma\delta R$, $\Sigma\delta r$, might be conveniently exhibited in columns contiguous to those which contain the corresponding integral numbers N, n, R, r.

It may be added that the principles of the calculation, as here given, are now as exact as might reasonably be required. The only hypothesis of an approximate nature is that the various fractional intervals at the commencement and at the termination of each year of age are assumed to be subject to the same rate of mortality as that which obtains for the entire year. If a recalculation were made the method of accurately apportioning these estimates, in accordance with the results previously found, would be too obvious to need any explanation here; but it can hardly ever be thought necessary to expend the resources of calculation upon such minute refinements.

Before concluding this paper it may not be out of place to make a few remarks on some of the existing methods of eradicating the irregularities of tabular results. Perhaps, the most prevalent assumption in these methods has been that of taking the arithmetical mean of a group of consecutive values and adopting the result as the adjusted value which belongs to the middle of the group. The practical operation of taking averages has an undoubted tendency to neutralize errors of a purely incidental kind; but unless the process be sound also in theory, it must at the same time be attended by the introduction of errors in principle. It is here proposed to inquire into the amount of these latter errors, which admit of being accurately calculated from a series of differences.

Referring to my paper on "Interpolation," vol. xi., page 83, "Central Formulæ," let the general term of a series of values be denoted by V_x and the differences as there stated, and we shall have, by the formula (γ),

$$V_n=(V)+(n-\tfrac{1}{2})a_1+\frac{(n-1)n}{2}\left((b)+\frac{n-\frac{1}{2}}{3}c_1\right)$$
$$+\frac{(n-2)(n-1)n(n+1)}{2.3.4}\left((d)+\frac{n-\frac{1}{2}}{5}e_1\right)$$
$$+\frac{(n-3)(n-2)(n-1)n(n+1)(n+2)}{2.3.4.5.6}\left((f)+\frac{n-\frac{1}{2}}{7}g_1\right)+\&c.$$

$$\therefore\ V_{n+1}=(V)+(n+\tfrac{1}{2})a_1+\frac{n(n+1)}{2}\left((b)+\frac{n+\frac{1}{2}}{3}c_1\right)$$
$$+\frac{(n-1)n(n+1)(n+2)}{2.3.4}\left((d)+\frac{n+\frac{1}{2}}{5}e_1\right)$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6}\left((f)+\frac{n+\frac{1}{2}}{7}g_1\right)+\&c.$$

$$V_{-n}=(V)-(n+\tfrac{1}{2})a_1+\frac{n(n+1)}{2}\left((b)-\frac{n+\frac{1}{2}}{3}c_1\right)$$
$$+\frac{(n-1)n(n+1)(n+2)}{2.3.4}\left((d)-\frac{n+\frac{1}{2}}{5}e_1\right)$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6}\left((f)-\frac{n+\frac{1}{2}}{7}g_1\right)+\&c.$$

$$\therefore\ V_{n+1}-V_{-n}=(2n+1)a_1+\frac{n(n+1).(2n+1)}{2.3}c_1$$
$$+\frac{(n-1)n(n+1)(n+2).(2n+1)}{2.3.4.5}e_1$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3).(2n+1)}{2.3.4.5.6.7}g_1+\&c.$$

Now, this must evidently be equal to the sum of the $2n+1$ intervening values of the first differences a, which lie between V_{-n} and V_{n+1}. Therefore, the arithmetical mean of these $2n+1$ values

$$=a_1+\frac{n(n+1)}{2.3}c_1+\frac{(n-1)n(n+1)(n+2)}{2.3.4.5}e_1$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6.7}g_1+\&c.$$

Hence, by analogy in the scheme of differences, we infer the general theorem that the arithmetical mean of $2n+1$ consecutive values of V

$$=V+\frac{n(n+1)}{2.3}b+\frac{(n-1)n(n+1)(n+2)}{2.3.4.5}d$$
$$+\frac{(n-2)(n-1)n(n+1)(n+2)(n+3)}{2.3.4.5.6.7}f+\&c. \quad . \; . \; (h),$$

where V, b, d, f, &c., are central values in the difference scheme. As n is integral the expression becomes completed in $n+1$ terms.

Taking the case $n=2$, we find for the mean of five values the simple formula $V+b+\frac{1}{5}d$.

Therefore the error engendered by assuming the arithmetical mean of five values to replace the middle value is equal to the middle second difference together with one-fifth of the fourth difference; and as fourth differences are usually small the second difference alone will in general be a sufficient indication of the amount of error.

The following example, with five logarithms, is given as a practical illustration :—

No.	Logarithms.	Differences.				
9	0·95424					
			+4576			
10	1·00000			−437		
			4139		+77	
11	1·04139	V		−360b		−20d
			3779		+57	
12	1·07918			−303		
			+3476			
13	1·11394					

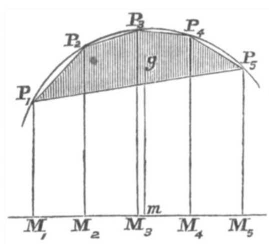
	5)5·18875		
Average of five values	1·03775	Second difference	−360
True middle value	1·04139	$\frac{1}{5}$ fourth difference	− 4
	−364	Error	−364

That the relation here stated is rigorously exact, whatever be the quantities, may be otherwise shown, as follows :—

The five values being denoted by V_{-2}, V_{-1}, V, V_{+1}, V_{+2}, the central second and fourth differences are

$$b= \qquad V_{-1}-2V+V_{+1}$$
$$d= \quad V_{-2} \ -4V_{-1}+6V-4V_{+1}+V_{+2}$$
$$\therefore b+\tfrac{1}{5}d=\tfrac{1}{5}(V_{-2}+V_{-1}+V+V_{+1}+V_{+2})-V$$
$$=\text{mean value}-\text{true middle value}.$$

The existance of the error here pointed out becomes very apparent when considered geometrically. In the annexed diagram let the five ordinates, P_1M_1, P_2M_2, &c., represent the five values. Join the five points, $P_1P_2P_3P_4P_5$, throughout the circuit by straight lines, so as to form an irregular pentagon, which, in the diagram, is distinguished by the shading; and let g be the mean centre of the five points, or centre of gravity of the pentagon.



Then, by geometry, it is well known that the arithmetical mean of the five ordinates is equal to the ordinate gm, demitted from the mean centre g of the pentagon, or of the five points P, and of course differs from the middle value P_3M_3. When the curve on which the points P are situated is concave downwards, as shown in the diagram, the error is obviously negative, and *vice versâ.*

In cases where the average is that of a group comprising a greater number of values than five, the magnitude of the error is of course greater, the formula (h) showing that it varies nearly as the square of the number of values that compose the group. When, therefore, as many as ten quantities are included, as in the formation of the English Life Table, and is necessarily so in everything derived from the population and registration returns for decennial periods, the discrepency involved by the stated assumption will amount to as much as four times the value of the central second difference.

It is not meant here to make any general exception to the simplifications obtainable by means of averages. A decided practical advantage, and a desirable reduction of casual imperfections, may often be gained by condensing an unadjusted series of results in groups; and such a process is, of course, perfectly legitimate, provided the several combinations are only represented to be what they really are, and treated accordingly. An instructive example of this kind may be found in the account of an investigation of the mortality experienced in the Eagle Insurance Company, given in vol. ii., page 199, in which Mr. Jellicoe, by grouping together the values of the mortality per cent. and discussing the combined

results with great skill and judgment, succeeds in distributing them into convenient tabular values, without any displacement of the aggregate mortality contained in each of the respective groups.

The methods of adjusting the probabilities of life by means of Gompertz's formula,

$$\log l_x = \log k + \log g . q^x,$$

are usually carried out by breaking the table up into portions, commencing and terminating at certain stated ages, and calculating each of these portions from a distinct set of constants. The results of all such calculations necessarily have the defect that the different component curves do not unite continuously at the points of juncture.*

In a former paper, "on Gompertz's Law of Mortality," I propounded certain generalizations under which a continuous curve might be generated, in the following manner:—"As the formula contains three arbitrary constants it will follow that if the calculated mortality be laid down in a curve, it may be made to pass through any three assigned points, and that the assumption of three such points will be sufficient to determine the values of the constants. But in a curve of mortality that does not conform with Gompertz's law, the computed values of the constants will vary with the position in which the three points are taken; and if the points be assumed to be indefinitely near to each other, the curves will osculate in their vicinity. Thus, in general, any curve of mortality may be considered as accurately represented by Gompertz's formula, if the symbols k, g, q, instead of being constants, are supposed to denote functions of the age x, and to appear as variable parameters that usually undergo but gradual changes in value." I have now to make the further suggestion that any curve of mortality may be accurately represented by Gompertz's formula if any one of those symbols is alone supposed to be a function of x and to appear as a variable parameter, the other two symbols being retained as constants throughout the curve. The conspicuous advantage derivable from Gompertz's formula when thus applied will be the mathematical fact that such parameter will in general not be subject to any capricious changes in value, and will therefore be convenient to deal with as a sort of key to the table of mortality.

Take the adjusted Experience Table as an example and let q be the variable parameter. Also, consulting the values contained

* It will be observed that Mr. Jellicoe, in his investigation just referred to, has not failed, with the exercise of tact and discrimination, to effect a practical remedy for the defective continuity observable at these points.

in the table, vol. x., page 124, let the assumed constants be $\log k = 5{\cdot}20000$, $\log g = -0{\cdot}20000$; so that the formula becomes

$$\log l_x = 5{\cdot}20000 - \tfrac{1}{5} q^x.$$

Then the values of the subsidiary number, $\log q$, may be found as follows:—

Age x	$\log l = \log k + \log g . q^x$	$\log g . q^x$	q^x	$x \log q$	$\log q$
10	5·0000000	−0·2000000	1·0000000	0·0000000	·0000000
15	4·9851389	0·2148611	1·0743055	0·0311278	·0020752
20	4·9697327	0·2302673	1·1513365	0·0612022	·0030601
25	4·9534456	0·2465544	1·2327720	0·0908827	·0036353
30	4·9359705	0·2640295	1·3201475	0·1206224	·0040207
35	4·9168801	0·2831199	1·4155995	0·1509404	·0043126
40	4·8957153	0·3042847	1·5214235	0·1822501	·0045563
45	4·8717772	0·3282228	1·6411140	0·2151387	·0047809
50	4·8420910	0·3579090	1·7895450	0·2527426	·0050549
55	4·8025617	0·3974383	1·9871915	0·2982397	·0054225
60	4·7479786	0·4520214	2·2601070	0·3541290	·0059021
65	4·6698188	0·5301812	2·6509060	0·4233943	·0065138
70	4·5543316	0·6456684	3·2283420	0·5089795	·0072711
75	4·3820170	0·8179830	4·0899150	0·6117143	·0081562
80	4·1235250	1·0764750	5·3823750	0·7309740	·0091372
85	3·7337588	1·4662412	7·3312060	0·8651755	·0101785
90	3·1202448	2·0797552	10·3987760	1·0169822	·0112998
95	1·9493900	−3·2506100	16·2530500	1·2109348	·0127467
100	$-\infty$	$-\infty$	∞	∞	∞

The resulting values of $\log q$, contained in the last column, judging by their progression, appear to be somewhat tractable.

Moreover, the principle originated by Gompertz also suggests another practical method of simplifying a discussion of the unadjusted results of a mortality table. Let $P_x = \log(\log l_x - \log l_{x+t})$ denote the logarithm of the logarithm of the reciprocal of the probability of a life, age x, surviving t years. Then since, according to Gompertz's law, a series of such values, viz. P_x, P_{x+t}, P_{x+2t}, &c., should be in arithmetical progression, we may at least infer that they are approximately so, and conclude that, in consequence of the smallness of their second differences, they must be a convenient set of numbers to have recourse to for the purposes of adjustment, so as to obtain a suitable continuity of progression in the final table. But, after the unexpected length to which this paper has already extended, I must forego any detailed discussion of what might be accomplished by the expedient here suggested, which would seem to have some promise of being not only interesting but of real practical utility.